Test particle transport in stochastic plasmas on the basis of the A-Langevin equations

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ABSTRACT

Trajectories of charged particles in a stochastic magnetic field can be described by the A-Langevin equations, which are second order stochastic differential equations. The mathematical formulation of the problem makes use of suitable assumptions for the stochastic properties of collisional events and magnetic fluctuations. We solve the A-Langevin equation and derive an expression for the velocity correlation function, which is averaged with respect to the stochastic variables. The resulting velocity correlator of the particle depends on the Lagrangian correlator of the magnetic field, which is estimated by applying an approximation method due to Corrsin. As the Lagrangian correlator is a function of the mean square displacement of the particles, an implicit differential equations arises for the description of the running diffusion coefficient. We investigated this equation in various cases and found different transport regimes, including the well-known Rechester-Rosenbluth result. The calculation generalizes previous results which have been based on the V-Langevin equations [1,2]. When the plasma consists of electrons and (heavy) dust particles, the latter can also be randomly charged.

I The A-Langevin Framework

A-Langevin equations are coupled second order stochastic differential equations for the space coordinates of a particle:

$$\frac{d^2}{dt^2}\boldsymbol{r}(t) = \frac{d}{dt}\boldsymbol{u}(t) = \frac{Ze}{mc}\boldsymbol{u}(t)\times\boldsymbol{B}(\boldsymbol{r}(t)) - \nu\,\boldsymbol{u}(t) + \boldsymbol{a}(t)$$

Stucture of the magnetic field:

$$\boldsymbol{B}(\boldsymbol{r}) \equiv B_0\,(b_0\,\boldsymbol{e}_z + \boldsymbol{b}(\boldsymbol{r}))$$

The Complete mathematical formulation needs assumptions of the stochastic properties for the stochastic magnetic field b(r,t), random collisions and the initial velocities:

- Initial velocities $\langle \boldsymbol{u}_0 \otimes \boldsymbol{u}_0 \rangle = v_t^2\,\mathbf{1}$
- Random collisions $\langle \boldsymbol{a}(t)\rangle = 0$ $\langle \boldsymbol{a}(t_1)\,\boldsymbol{a}(t_2)\rangle = \mathbf{1}_\Sigma\,A\,\delta(t_1 - t_2)$
- Fluctuating b-Field $\langle \boldsymbol{b}(t)\rangle = 0$ $\langle \boldsymbol{b}(t_1)\otimes\boldsymbol{b}(t_2)\rangle = \beta^2\,\mathcal{L}(t)$

Solution of the A-Langevin equations:

- Transformation to **Brownian structure**
- Solution by a **Green Function Method**

In terms of the **Green propagator** *G(t,t‘)*, the solution is given by:

$$\boldsymbol{u}(t) = e^{-i\Omega_0 b_0 L_3 t - \nu t}\,G(t)\,\boldsymbol{u}_0 + e^{-i\Omega_0 b_0 L_3 t - \nu t}\int_0^t G(\tau, t)\,e^{i\Omega_0 b_0 \tau + \nu\tau}\,\boldsymbol{a}(\tau)\,d\tau$$

The solution leads to the **velocity correlation function**:

$$\boldsymbol{u}(t_1)\otimes\boldsymbol{u}(t_2) = e^{-i\Omega_0 b_0 L_3 t_1 - \nu t_1}\,G(t_1)\,\boldsymbol{u}_0 \otimes e^{-i\Omega_0 b_0 L_3 t_2 - \nu t_2}\,G(t_2)\,\boldsymbol{u}_0 +$$
$$+ e^{-i\Omega_0 L_3 t_1 - \nu t_1}\int_0^{t_1} G(\tau, t_1)\,e^{i\Omega_0 b_0 \tau + \nu\tau}\,\boldsymbol{a}(\tau)\,d\tau \otimes e^{-i\Omega_0 b_0 L_3 t_2 - \nu t_2}\int_0^{t_2} G(\tau, t_2)\,e^{i\Omega_0 b_0 \tau + \nu\tau}\,\boldsymbol{a}(\tau)\,d\tau$$

II Averaging the Velocity Correlation Function

The solution contains stochastic variables and has to be averaged with respect to the stochastic data defined in section I. Several calculation steps are necessary for averaging:

- Integration of the collisional correlation function
- **Second cumulant expansion**

$$\langle\langle\langle u^i(t_1)\,u^i(t_2)\rangle_{u_0}\rangle_a\rangle_b = v_t^2\,e^{-\nu\tau}\begin{cases} e^{-i\Omega_0 b_0\tau}\,e^{-f(\tau)}\,e^{-\gamma(\tau)} & \{x, y\} \\ e^{-2\,\mathrm{Re}[f(\tau)]} & \{z\}\end{cases}$$

$$f(t) \equiv \Omega_0^2\,\beta^2\int_0^\tau (t-\tau)\,e^{i\Omega_0 b_0\tau}\,\mathcal{L}_\perp(\tau)\,d\tau$$

$$\gamma(t) \equiv \Omega_0^2\,\beta^2\int_0^\tau (t-\tau)\,\mathcal{L}_z(\tau)\,d\tau$$

The expressions can be simplified by integrating *f*(t) and *g*(t) using a spectral approximation within the Fourier space:

$$\langle u\,u\rangle_\perp = v_t^2\,e^{-\nu t}\,e^{-i\Omega_0 b_0 t - i\beta^2\Omega_0 t}\,(1 - 3\beta^2 + 3\beta^2\,\mathcal{L}_z(t) - \ldots + \ldots) *$$
$$* \left(1 - \beta^2 + \beta^2\,\mathcal{L}_\perp(t)\,e^{i\Omega_0 b_0 t}\,e^{-2\beta^2 i\Omega_0 b_0 t} - \ldots + \ldots\right)$$

$$\langle u\,u\rangle_{||} = (1-\beta^2)\,v_t^2\,e^{-\nu t} + 2\beta^2\,v_t^2\,e^{-\nu t}\,\mathcal{L}_\perp(t)\cos(\Omega_0 t)$$

III Lagrangian Correlation Function

With the test particle trajectory still unknown the **Lagrangian correlator** has to be derived from the **Eulerian correlation** function of the magnetic field:

$$\mathcal{E}[\boldsymbol{r}(t)] = \begin{pmatrix} 1 - \frac{y^2}{\lambda_\perp^2} & 0 \\ 0 & 1 - \frac{x^2}{\lambda_\perp^2}\end{pmatrix}\exp\left(-\frac{x^2+y^2}{2\lambda_\perp} - \frac{z^2}{2\lambda_{||}^2}\right)$$

↓

Corrsin Approximation [5]

$$\mathcal{L}(t) = \int d\boldsymbol{r}\,\langle\langle\mathcal{E}(\boldsymbol{r})\,\delta(\boldsymbol{r} - \delta\boldsymbol{r}(t))\rangle\rangle$$

$$\gamma \equiv \langle\delta(x - \zeta(t))\rangle = (2\pi)^{-3}\int d^3k\,e^{i\boldsymbol{k}\boldsymbol{r}}\,\langle e^{-i\boldsymbol{k}\,\delta\boldsymbol{r}}\rangle$$

↓

$$\mathcal{L} = \left(1 + \frac{\delta z^2}{\lambda_{||}^2}\right)^{-1/2}\left(1 + \frac{\delta x^2}{\lambda_\perp^2}\right)^{-i} \qquad i = \begin{cases} 2 & \perp \\ 1 & || \end{cases}$$

IV The A-MSD-Equations

IV.a Classical Diffusion

Renormalizing the phase pertubations and using the **Green-Kubo-relation**, an equation system for the Mean-Squared-Deviation can be derived:

$$\langle\delta r_i^2(t'')\rangle = 2\int_0^{t''}\int_0^{t'}\langle\langle\langle u^i(t)\,u^i(0)\rangle_{u_0}\rangle_a\rangle_b\,dt\,dt' \qquad i = x, y, z.$$

↓

$$\frac{d^2}{dt^2}\langle\delta x^2\rangle = v_t^2\,e^{-\nu t}\,e^{-i\Omega_0 t} + v_t^2\,\beta^2\,e^{-\nu t}\,\mathcal{L}_\perp[\langle\delta x^2(t)\rangle, \langle\delta z^2(t)\rangle]$$

$$\frac{d^2}{dt^2}\langle\delta z^2\rangle = v_t^2\,e^{-\nu t} + 2\beta^2\,v_t^2\,e^{-\nu t}\,\mathcal{L}_\perp[\langle\delta x^2(t)\rangle, \langle\delta z^2(t)\rangle]\cos(\Omega_0 t)$$

The leading order terms correspond to **classical collisional** diffusion and reproduce the expressions for the diffusion coefficients:

$$\langle\delta x^2\rangle^{(0)} = \frac{v_t^2}{\Omega_0^2}\left\{1 + \nu t - \left[\cos(\Omega_0 t) + \frac{2\nu}{\Omega_0}\sin(\Omega_0 t)\right]e^{-\nu t}\right\}$$

$$\langle\delta z^2\rangle^{(0)} = \frac{v_t^2}{\nu^2}\,(\nu t - 1 + e^{-\nu t})$$

$$\chi_\perp \equiv \frac{v_t^2\,\nu}{2\,\Omega_0^2} \qquad \chi_{||} \equiv \frac{v_t^2}{2\nu}$$

IV.b The quasilinear limit and the Kadomtsev-Pogutse case

The **anomalous diffusion** is caused by the higher order terms. In the collisionless limit, the well-known **quasilinear diffusion** [1,3] coefficient is found using a perturbative treatment in β and $\lambda_\perp \to \infty$ assuming:

$$\chi_\perp^{(0)(QL)} = v_t^2\,\beta^2\int_0^{\lambda_{||}/v_t}\frac{1}{\sqrt{1 + \frac{v_t^2 t^2}{\lambda_{||}^2}}}\,dt = v_t\,\beta^2\,\lambda_{||}\,\mathrm{arcsinh}(1)$$

$$\chi_\perp^{(1)(QL)} = -4\left(\sqrt{2} - 1\right)\frac{\beta^4\,\lambda_{||}^3}{\lambda_\perp^2}$$

The last expression shows excellent agreement with the work of Wang [1], though derived in a quite different way. In the case of an inifinite parallel correlation length, the **Kadomtsev-Pogutse** diffusion coefficient is found [6,8].

IV.c Rechester-Rosenbluth case

The analysis of the complete differential equation system is rather complicated and shows in the strongly collisional limit the celebrated **Rechester-Rosenbluth**-coefficient with its characteristic reduction by the **Kolmogorov length** L_K:

$$\chi_\perp = c_1\,\frac{\chi_{||}\,\lambda_\perp^2}{L_K^2}\,\frac{1}{\log^2\left(c_2\,\frac{2}{\pi}\,\frac{\chi_{||}\,\lambda_\perp^2}{\chi_\perp\,L_K^2}\right)}$$

IV.d Zero mean field limit

The assumption of a vanishing mean field B=0 leads to equal correlation functions in all three space dimensions:

$$\frac{d^2}{dt^2}\langle\delta r_i^2\rangle = \{2 - 6\beta^2 + 6\beta^2\,\mathcal{L}(t)\}\,v_t^2\,e^{-\nu t}\,e^{-i\beta^2\Omega_0 t} \qquad i = x, y, z$$

This situation **cannot** be described consistently in other approaches like the **V-Langevin method** [1,2] and the MDIA [2]. Numerical investigations of parallel and perpendicular transport [9] for vanishing guiding fields agree with the predictions of the A-Langevin model.

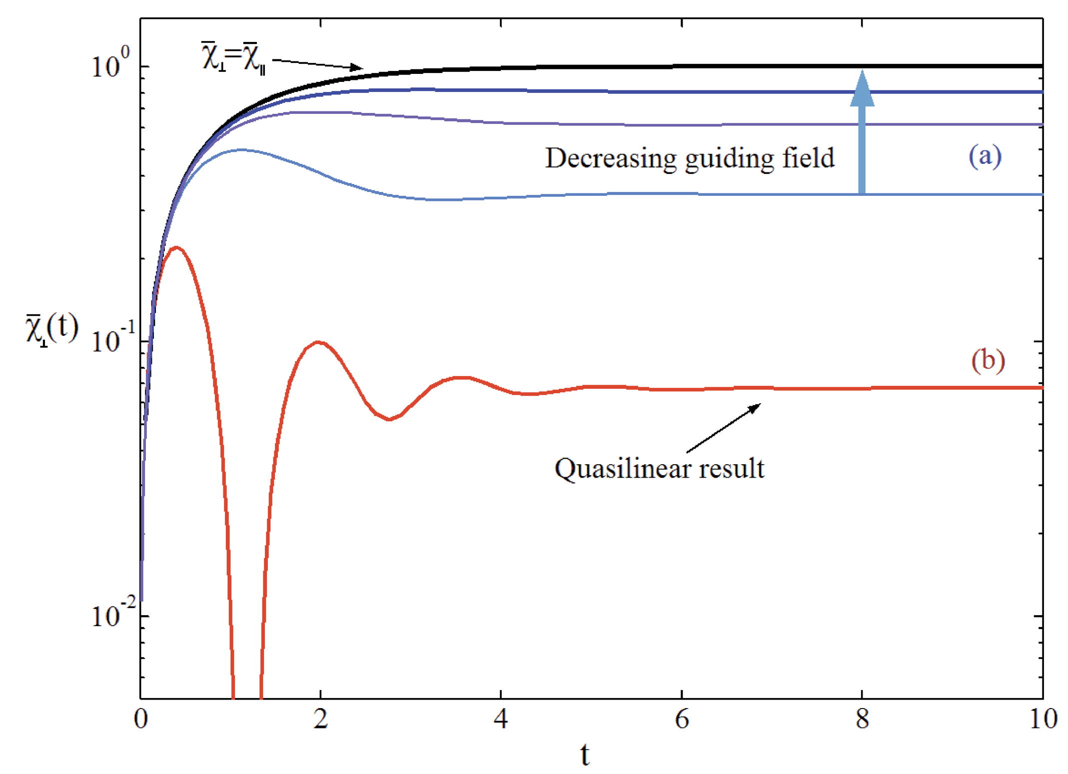

Fig.1 Dimensionless perpendicular diffusion coefficient for different parameter regimes. (a) Reducing the guiding field leads to the coincidence of parallel and perpendicular transport coefficients. (b) The well-known quasilinear diffusion.

V Summary

- The solution and the velocity correlation function of a test-particle were derived from the **A-Langevin**-equation
- The correlation function was averaged with respect to the initial velocities, collisions and magnetic field fluctuations
- On the basis of **Corrsin**´s independence hypothesis, the **Lagrangian correlation function** was connected to the **Eulerian correlator** of the magnetic field
- A set of differential equations was derived, which reproduces the well-known diffusion regimes, including the **Rechester-Rosenbluth** case
- The A-Langevin approach extends the description to cases, where other methods fail, e.g. the zero mean field limit

Bibliography

[1] H.D.Wang, R.Balescu et.al., *Diffusive processes in a stochastic magnetic field*, Phys. Rev. E **51** (5) 4844 (1995)
[2] E.Vanden-Eijnden, R.Balescu, *Statistical description and trans port in stochastic magnetic fields*, Phys. Plasmas **3** (3) 874 (1996)
[3] A.B.Rechester and M.N.Rosenbluth, Phys.Rev.Lett. 40 38 (1978)
[4] R.Balescu, H.D.Wang, J.H.Misguich, Phys.Plasmas 1 3826 (1994)
[5] S.Corrsin, in Atmospheric diffusion and air pollution, edited by F. N. Frenkiel, P.A.Sheppard (Academic, New York, 1990) 161
[6] B.B.Kadomtsev, O.P.Pogutse, in *Plasma Physics and Controlled Nuclear Fusion Research*, Proceedings of the 7th International Conference, Insbruck, 1978
[7] B.B.Kadomtsev, O.P.Pogutse, Phys.Rev.Lett. **25** 1155 (1970)
[8] M.B.Isichenko, Plasma Phys. Controlled Fusion **33** 795 (1991)
[9] F.Casse, M.Lemoine, G.Pelletier, Phys.Rev.D 65 023002 (2002)